

PACS 42.25.Dd, 42.25.Ja, 42.81.-i

Matrix model of inhomogeneous medium with circular birefringence in single scattering case

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Abstract. The paper is devoted to the analysis of light scattering by inhomogeneous circular birefringent media in a single scattering case. The object under investigation is a circular birefringent crystalline slab with surface inhomogeneity. For the analysis, we derive the Mueller matrix model for this media and use Cloude's coherency matrix method. Sample calculations are given for quartz SiO_2 and paratellurite TeO_2 .

Keywords: light scattering, depolarization, Mueller matrix, circular birefringence, Cloude's coherency matrix, entropy.

Manuscript received 04.03.09; accepted for publication 18.03.09; published online 20.03.09.

1. Introduction

Interaction of medium with a beam of polarized light can be described as transformation of the input Stokes parameters into output ones by the Mueller (scattering) matrix that contains information related to the anisotropic and depolarization properties of the medium. Consequently, a detailed interpretation of the Mueller matrix elements may be used to characterize the physical properties of studied medium [2–4].

In spite of the fact that typically the circular birefringence effect is about 10^3 - 10^5 times smaller than that for linear birefringence, polarization of radiation is affected much more than that expected for this small value. The Mueller matrix of circular birefringence is [4]:

$$\mathbf{M}^{CB}(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(2\varphi) & \sin(2\varphi) & 0 \\ 0 & -\sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (1)$$

where φ is the value of circular birefringence.

In this paper, we derive the Mueller matrix of the circular birefringent crystalline slab with surface inhomogeneity and analyze information on medium anisotropy and depolarization properties containing in the elements of derived matrix using an additive model basing on the so-called Cloude coherency matrix [5, 6].

2. Mueller matrix model

As shown earlier [1], the Mueller matrix of elliptically birefringent crystalline slab with surface inhomogeneity (roughness) in the eigen coordinate system has the following form:

$$\mathbf{M}^{eigen} = \begin{pmatrix} 1 & \frac{\Phi_{11} - \Phi_{22}}{\Phi_{11} + \Phi_{22}} & 0 & 0 \\ \frac{\Phi_{11} - \Phi_{22}}{\Phi_{11} + \Phi_{22}} & 1 & 0 & 0 \\ 0 & 0 & \frac{\Phi_{12} + \Phi_{21}}{\Phi_{11} + \Phi_{22}} & i \frac{\Phi_{12} - \Phi_{21}}{\Phi_{11} + \Phi_{22}} \\ 0 & 0 & -i \frac{\Phi_{12} - \Phi_{21}}{\Phi_{11} + \Phi_{22}} & \frac{\Phi_{12} + \Phi_{21}}{\Phi_{11} + \Phi_{22}} \end{pmatrix}, \quad (2)$$

where

$$\Phi_{uv} = \Phi_{uv}^{brf} \left\{ (1 - \eta_{uv}) \exp \left\{ - \left(\frac{kpw}{2z} \right)^2 \right\} + \right. \quad (2a)$$

$$\left. + 1 \frac{\eta_{uv}}{\sigma_{uv}^2 w^2 + 1} \exp \left\{ - \frac{1}{\sigma_{uv}^2 w^2 + 1} \left(\frac{kpw}{2z} \right)^2 \right\} \right\},$$

$$\Phi_{uv}^{brf} = \exp \left(ik(n_u - n_v) \bar{h} - k^2 \sigma_h^2 (n_u - n_v)^2 / 2 \right), \quad (2b)$$

$$w = d \left(1 + 4z^2 / d^4 k^2 \right)^{1/2}, \quad (2c)$$

$$\sigma_{uv}^2 = k^2 \sigma_h^2 (n_u - 1)(n_v - 1), \quad (2d)$$

$$\eta_{uv} = 1 - \exp(-\sigma_{uv}^2), \quad (2e)$$

$k = 2\pi/\lambda$ is the wave number; \bar{h} is the thickness of a crystalline slab; n_u and n_v denotes refractive indexes of a medium associated with its elliptical eigenpolarizations, and d is the beam radius.

If to assume that the Mueller matrix Eq. (2) is presented in circular eigen coordinate system, i.e. $n_u = n_r$ and $n_v = n_l$ denotes in this case the refractive indexes associated with circular, right and left, eigenpolarizations, then this matrix corresponds to a matrix model of the medium characterized by circular birefringence at least in a given direction of light propagation.

If the Mueller matrix in eigencoordinate system, \mathbf{M}^{eigen} , Eq. (2) is known, then the Mueller matrix in the laboratory system, \mathbf{M}^{lab} , can be found using a well known algebra [7] with the corresponding transformation matrix associated with circular eigenvectors:

$$\mathbf{E}_{e1,e2} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}. \quad (3)$$

Then, the Mueller matrix in the laboratory coordinate system takes the form:

$$\mathbf{M}^{lab} = \begin{pmatrix} 1 & 0 & 0 & \frac{\Phi_{11}-\Phi_{22}}{\Phi_{11}+\Phi_{22}} \\ 0 & \frac{\Phi_{12}+\Phi_{21}}{\Phi_{11}+\Phi_{22}} & i \frac{\Phi_{12}-\Phi_{21}}{\Phi_{11}+\Phi_{22}} & 0 \\ 0 & -i \frac{\Phi_{12}-\Phi_{21}}{\Phi_{11}+\Phi_{22}} & \frac{\Phi_{12}+\Phi_{21}}{\Phi_{11}+\Phi_{22}} & 0 \\ \frac{\Phi_{11}-\Phi_{22}}{\Phi_{11}+\Phi_{22}} & 0 & 0 & 1 \end{pmatrix}, \quad (4)$$

3. Analysis of the derived matrix model

Next, we examine our model with the widely used crystals: quartz SiO_2 and paratellurite TeO_2 with elliptical eigenpolarizations, in a general case. The analysis is carried out for exact forward scattering in the case when light propagation occurs in the direction where both crystals exhibit only circular birefringence.

Comparing the structures of the Mueller matrices for a slab of circular birefringent medium without inhomogeneity Eq. (1) and for that with inhomogeneity Eq. (4), one can see that, because of elements m_{14} and m_{41} of the matrix Eq. (4) are generally not equal to zero, the presence of inhomogeneity results in occurrence of new polarization and depolarization effects.

3.1. The case of $\sigma_h = 0$

From Eqs. (2) and (4), setting $\sigma_h = 0$, we get:

$$\mathbf{M}_{\sigma_h=0}^{lab} = \frac{2}{1 + \frac{z^2 \lambda^2}{d^4 \pi^2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{2\bar{h}\pi\Delta n}{\lambda}\right) & -\sin\left(\frac{2\bar{h}\pi\Delta n}{\lambda}\right) & 0 \\ 0 & \sin\left(\frac{2\bar{h}\pi\Delta n}{\lambda}\right) & \cos\left(\frac{2\bar{h}\pi\Delta n}{\lambda}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5)$$

The Mueller matrix Eq. (5) is a matrix of circular birefringence which is equivalent to the matrix Eq. (1). Note that when one sets $\bar{h} = 0$, the matrix Eq. (5) becomes identity.

3.2. The case of large inhomogeneities

In [1] it was shown that when inequality

$$\sigma_h > \frac{\lambda\sqrt{7}}{2\pi|n_r - n_l|} \quad (6)$$

is satisfied, then the matrix Eq. (2) contains only four non-zero elements, these elements form 2×2 principal minor, and is singular. This is the so-called case of large inhomogeneities.

The Mueller matrix in laboratory coordinate system Eq. (4) in this case takes the following form:

$$\mathbf{M}_{\sigma_h \rightarrow \infty}^{lab} = \begin{pmatrix} 1 & 0 & 0 & \frac{\Phi_{11}-\Phi_{22}}{\Phi_{11}+\Phi_{22}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\Phi_{11}-\Phi_{22}}{\Phi_{11}+\Phi_{22}} & 0 & 0 & 1 \end{pmatrix}. \quad (7)$$

Table presents the values of inhomogeneity σ_h for which the inequality Eq. (6) is satisfied and the Mueller matrix has the form Eq. (7) for paratellurite TeO_2 and quartz SiO_2 crystals. Estimations are derived for the wavelength $\lambda = 0.63\mu\text{m}$.

The derived estimations show that at the given wavelength the case of large inhomogeneity is not actually attainable and, thus, in the range of inhomogeneity $\sigma_h < d$ the Mueller matrix of inhomogeneous circularly birefringent medium Eq. (2) is not singular.

In a general case, from Eq. (4) for the output intensity we have:

$$I^{out} = s_1^{out} = 1 + \frac{\Phi_{11}-\Phi_{22}}{\Phi_{11}+\Phi_{22}} \sin(2\varepsilon^{inp}). \quad (1)$$

Table. Estimations of acceptability of approximations Eq. (6) for paratellurite and quartz crystals at the wavelength $\lambda=0.63\mu\text{m}$.

Crystals	n_r	n_l	σ_h , m
TeO_2	2.2600	2.2594	0.0004
SiO_2	1.5427	1.5426	0.0020

So, the output intensity depends only on the value of ellipticity ε^{inp} of input polarization. In this case, maximum and minimum values of the output intensity are

$$I_{\max}^{lab} = \frac{2\Phi_{11}}{\Phi_{11} + \Phi_{22}}, \quad (9)$$

$$I_{\min}^{lab} = \frac{2\Phi_{22}}{\Phi_{11} + \Phi_{22}}, \quad (10)$$

and are achieved at $\varepsilon^{inp} = \pm\pi/4$, respectively.

The degree of polarization of output radiation is

$$p^{out} = \frac{\sqrt{(s_2^{out})^2 + (s_3^{out})^2 + (s_4^{out})^2}}{s_1^{out}} = \frac{\sqrt{((\Phi_{11} - \Phi_{22}) + (\Phi_{11} + \Phi_{22}) \sin(2\varepsilon^{inp}))^2 + 4\Phi_{12}\Phi_{21} \cos^2(2\varepsilon^{inp})}}{((\Phi_{11} + \Phi_{22}) + (\Phi_{11} - \Phi_{22}) \sin(2\varepsilon^{inp}))}. \quad (11)$$

Then $p_{\max}^{\min} = 1$, i.e. the case when output radiation is completely polarized, is achieved at $\varepsilon^{inp} = \pi/4$, and at $\varepsilon^{inp} = -\pi/4$ we have $p^{out} = \frac{\Phi_{11}}{\Phi_{22}}$. The minimum value of degree of polarization is

$$p_{\min}^{out} = \frac{\sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} + \Phi_{22}}, \quad (12)$$

and is achieved at $\varepsilon = 0^0$.

Fig. 1 shows the dependences of minimum values of polarization degree for output radiation p_{\min}^{out} as a function of the inhomogeneity σ_h and wavelength λ .

Let us estimate the accessible minimum value of p_{\min}^{out} . This value corresponds to the case of large inhomogeneity $\sigma_h \rightarrow \infty$. Then, using the explicit form of the Mueller matrix for this case Eq. (7), we get:

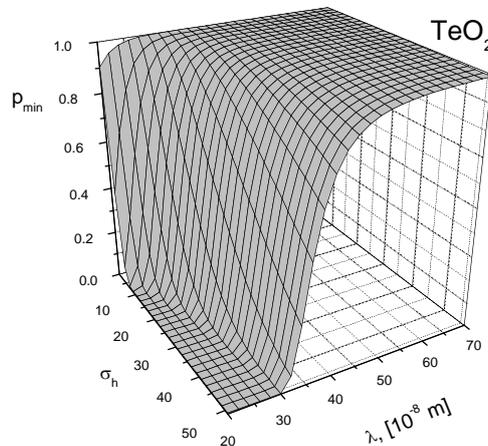
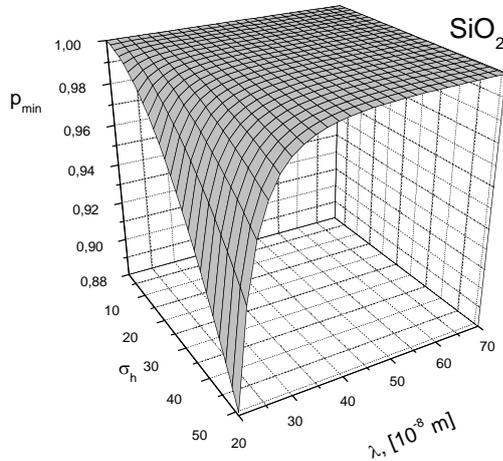


Fig. 1. Dependences of p_{\min}^{out} on the value of inhomogeneity σ_h and wavelength λ : (a) quartz SiO_2 ; (b) paratellurite TeO_2 .

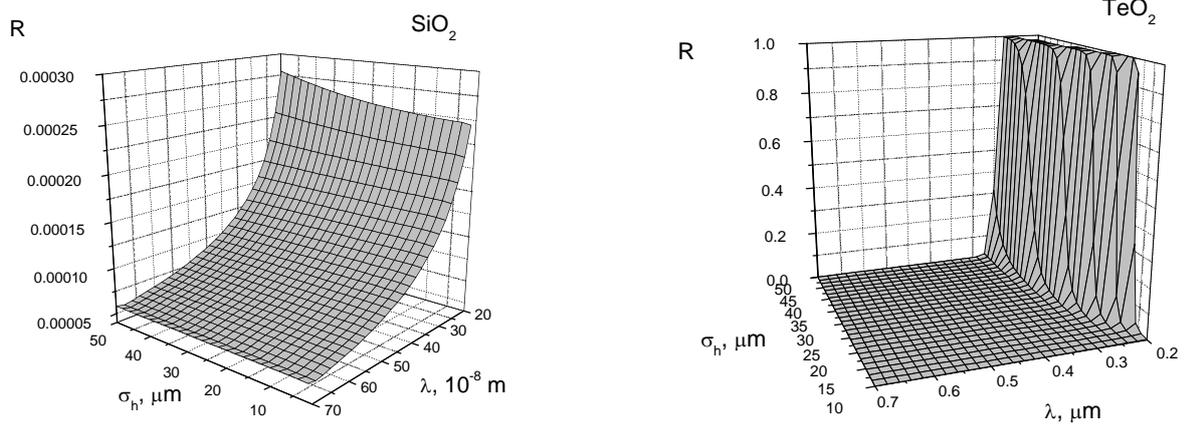


Fig. 2. Dependences of the values of circular dichroism R on inhomogeneity σ_h and wavelength λ : (a) quartz SiO_2 ; (b) paratellurite TeO_2 .

$$\lambda_2 = 1 - \frac{\sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} + \Phi_{22}}, \quad (16)$$

$$\lambda_3 = 0, \quad \lambda_4 = 0 \quad (17)$$

and eigenvectors:

$$\xi_1 = \begin{pmatrix} \frac{\Phi_{12} + \Phi_{21} + \sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} - \Phi_{22} - \Phi_{12} + \Phi_{21}} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (18)$$

$$\xi_2 = \begin{pmatrix} \frac{\Phi_{12} + \Phi_{21} - \sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} - \Phi_{22} - \Phi_{12} + \Phi_{21}} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (19)$$

$$\xi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \quad (20)$$

The Jones matrices associated with eigenvalues ξ_1 and ξ_2 Eqs. (18) and (19), which correspond to two non-zero eigenvalues λ_1 and λ_2 , Eqs. (15) and (16), are:

$$\mathbf{J}_1 = \begin{pmatrix} \frac{\Phi_{12} + \Phi_{21} + \sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} - \Phi_{22} - \Phi_{12} + \Phi_{21}} & -i \\ i & \frac{\Phi_{12} + \Phi_{21} + \sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} - \Phi_{22} - \Phi_{12} + \Phi_{21}} \end{pmatrix}. \quad (21)$$

$$\mathbf{J}_2 = \begin{pmatrix} \frac{\Phi_{12} + \Phi_{21} - \sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} - \Phi_{22} - \Phi_{12} + \Phi_{21}} & -i \\ i & \frac{\Phi_{12} + \Phi_{21} - \sqrt{(\Phi_{11} - \Phi_{22})^2 + 4\Phi_{12}\Phi_{21}}}{\Phi_{11} - \Phi_{22} - \Phi_{12} + \Phi_{21}} \end{pmatrix}. \quad (22)$$

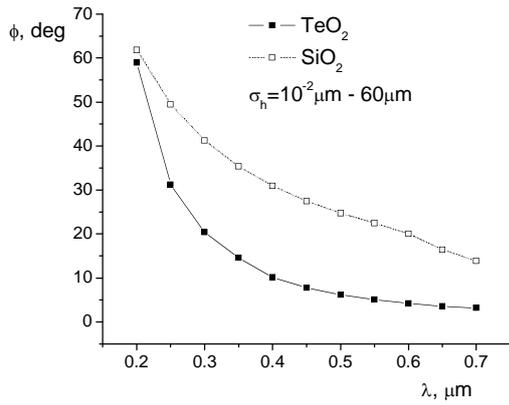


Fig. 3. Dependences of the values of circular birefringence ϕ on inhomogeneity σ_h and wavelength λ .

It can be seen that the values of circular birefringence ϕ do not depend on inhomogeneity σ_h , and its behavior versus wavelength λ is completely determined by the values of refractive indexes and their variation with changing λ .

The fact that, with the exception of R and ϕ , all other types of anisotropy are absent enables to build the analytical expressions for the values of circular dichroism R and circular birefringence ϕ . For that, we use the so-called second Jones' equivalence theorem [9], according to which the Jones matrix of the medium characterized only by circular dichroism and circular birefringence, \mathbf{J}^{CBD} , can be obtained in the following way:

$$\mathbf{J}^{CBD} = \mathbf{J}^{CB} \mathbf{J}^{CD} = \begin{pmatrix} \cos(\phi) + iR\sin(\phi) & -iR\cos(\phi) + \sin(\phi) \\ iR\cos(\phi) - \sin(\phi) & \cos(\phi) + iR\sin(\phi) \end{pmatrix}. \quad (23)$$

Introducing evident representation of the Jones matrix associated with the maximum eigenvalue Eq. (21) in the form:

$$\mathbf{J}^{CBD} = \begin{pmatrix} j_{11} & -i \\ i & j_{11} \end{pmatrix}, \quad (2)$$

and making element wise comparison of the matrices Eqs. (23) and (24), for values R and ϕ we get

$$R = \frac{1 + \text{Im}^2(j_{11}) + \text{Re}^2(j_{11}) - \sqrt{\text{Im}^4(j_{11}) + 2\text{Im}^2(j_{11})(\text{Re}^2(j_{11}) - 1) + (\text{Re}^2(j_{11}) + 1)^2}}{2\text{Im}(j_{11})}, \quad (25)$$

$$\phi = \arccos \left(\sqrt{\frac{1 + \text{Im}^4(j_{11}) + 2\text{Re}^2(j_{11}) + \text{Re}^4(j_{11}) + t(1 - \text{Re}^2(j_{11}) - \text{Im}^2(j_{11})) - 2\text{Im}^2(j_{11})(1 - \text{Re}^2(j_{11}))}{2t^2}} \right), \quad (26)$$

$$\text{where } t = \sqrt{\text{Im}^4(j_{11}) + 2\text{Im}^2(j_{11})(\text{Re}^2(j_{11}) - 1) + (\text{Re}^2(j_{11}) + 1)^2}.$$

4. Conclusion

In summary, we have studied light scattering by inhomogeneous circular birefringent media in single scattering case. We derive the Mueller matrix model for this class of media and show that the resulting Mueller matrix for given observation angle implies the presence of circular dichroism, whereas, initial anisotropy of studied medium is purely circular birefringence. We show that for this medium the minimum value of polarization degree p_{\min}^{out} is determined only by the values of refractive indexes at a given wavelength. Thus, the value p_{\min}^{out} cannot be equal to zero.

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